LEARNING MODELS

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CEPREMAP

Learning models

Two examples:

- R. Edge, T. Laubach and J. Williams (2004) “Learning and Shifts in Long–Run Productivity Growth” (ELW)
- T. Sargent, N. Williams and T. Zha (2005) “Shocks to Government Beliefs: The Rise and Fall of American Inflation” (SWZ)

Learning

Learning may be done by

- private agents (ELW)
- government (SWZ)

Learning may be about

- hidden state (ELW)
- parameters value (SWZ)

Learning may be done via

- recursive least squares
- Kalman filtering (ELW and SWZ)

Solving learning models

- In each period, agents solve a new decision problem.
- The model that they use to solve this problem isn't the true law of motion of the economy.

ELW model

- Two sector neoclassical growth model
- Endogenous labor supply
- Permanent shocks to growth rate of productivity
- Authors use a projection method to solve the model

Production technology

$$
\begin{aligned}
C_t &= A_t^{1-\alpha} K_{c,t}^{\alpha} L_{c,t}^{1-\alpha} \\
I_t &= (Z_t A_t)^{1-\alpha} K_{i,t}^{\alpha} L_{i,t}^{1-\alpha}
\end{aligned}
$$

with C_t, production of consumption goods, I_t, production of investment goods, A_t, aggregate technology process common to both sectors, Z_t technology process specific to the investment good sector. $K_{j,t}$, stock of capital in sector j, at the end of period t, $L_{j,t}$, labor employed in sector j.

Accumulation

$$
\begin{aligned}
K_t &= (1-\delta)K_{t-1} + I_t \\
K_{t-1} &= K_{c,t} + K_{i,t}
\end{aligned}
$$

Capital is freely mobile between the two sectors, after learning the productivity shocks at the beginning of the period.

Households

$$\max \mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t U_t$$

$$U_t = \begin{cases} \ln \frac{C_t}{H_t} + \zeta \ln\left(1 - \frac{L_t}{H_t}\right) & \text{for } \gamma = 1 \\ \frac{1}{1-\gamma}\left[\frac{C_t}{H_t}\left(1 - \frac{L_t}{H_t}\right)^{\zeta}\right]^{1-\gamma} & \text{otherwise} \end{cases}$$

where H_t is the household's size.

$$L_t = L_{i,t} + L_{c,t}$$

First order conditions

$$\frac{K_{i,t}}{K_{c,t}} = \frac{L_{i,t}}{L_{c,t}}$$

$$\left(\frac{C_t}{H_t}\right)^{-\gamma}\left(1-\frac{L_t}{H_t}\right)^{\zeta(1-\gamma)} Z_t^{-(1-\alpha)}$$

$$= \beta\mathcal{E}_t\left(\frac{C_{t+1}}{H_{t+1}}\right)^{-\gamma}\left(1-\frac{L_{t+1}}{H_{t+1}}\right)^{\zeta(1-\gamma)}\left(\alpha\frac{C_{t+1}}{K_{c,t+1}}+(1-\delta)Z_{t+1}^{-(1-\alpha)}\right)$$

$$\zeta\frac{L_{c,t}}{H_t} = (1-\alpha)\left(1-\frac{L_t}{H_t}\right)$$

Growth processes

$$
\begin{aligned}
\ln A_t &= \ln A_{t-1} + \ln G_{A,t} + \epsilon_{A,t} \\
\ln G_{A,t} &= \ln G_{A,t-1} + \nu_{A,t} \\
\ln Z_t &= \ln Z_{t-1} + \ln G_{Z,t} + \epsilon_{Z,t} \\
\ln G_{Z,t} &= \ln G_{Z,t-1} + \nu_{Z,t} \\
H_t &= G_H H_{t-1}
\end{aligned}
$$

Stationary variables

$$
\begin{aligned}
\ell_t &= \frac{L_t}{H_t} \\
c_t &= \frac{C_t}{H_t A_t Z_t^\alpha} \\
i_t &= \frac{I_t}{H_t A_t Z_t} \\
k_t &= \frac{K_t}{H_t A_t Z_t}
\end{aligned}
$$

Stationarized model

$$
\begin{aligned}
\frac{k_{i,t}}{k_{c,t}} &= \frac{\ell_{i,t}}{\ell_{c,t}} \\
c_t^{-\gamma}\left(1-\ell_t\right)^{\zeta(1-\gamma)} &= \beta \mathcal{E}_t \left(c_{t+1} G_{A,t+1} e^{\epsilon_{A,t+1}} \left(G_{Z,t+1} e^{\epsilon_{Z,t+1}}\right)^{\alpha}\right)^{-\gamma} \\
& \quad \left(1-\ell_{t+1}\right)^{\zeta(1-\gamma)} \left(\alpha \frac{c_{t+1}}{k_{c,t+1}} + 1 - \delta\right) \left(G_{Z,t+1} e^{\epsilon_{Z,t+1}}\right)^{-(1-\alpha)} \\
\zeta \ell_{c,t} &= (1-\alpha)\left(1-\ell_t\right) \\
c_t &= k_{c,t}^{\alpha} \ell_{c,t}^{1-\alpha} \\
i_t &= k_{i,t}^{\alpha} \ell_{i,t}^{1-\alpha} \\
k_t &= (1-\delta) \frac{k_{t-1}}{G_H G_{A,t} e^{\epsilon_{A,t}} G_{Z,t} e^{\epsilon_{Z,t}}} + i_t \\
\frac{k_{t-1}}{G_H G_{A,t} e^{\epsilon_{A,t}} G_{Z,t} e^{\epsilon_{Z,t}}} &= k_{c,t} + k_{i,t}
\end{aligned}
$$

Learning model

$$
\begin{aligned}
\frac{k_{i,t}}{k_{c,t}} &= \frac{\ell_{i,t}}{\ell_{c,t}} \\
c_t^{-\gamma}\left(1-\ell_t\right)^{\zeta(1-\gamma)} &= \beta\mathcal{E}_t\left(c_{t+1}\widehat{G}_{A,t+1}e^{\epsilon_{A,t+1}}\left(\widehat{G}_{Z,t+1}e^{\epsilon_{Z,t+1}}\right)^{\alpha}\right)^{-\gamma} \\
&\quad \left(1-\ell_{t+1}\right)^{\zeta(1-\gamma)}\left(\alpha\frac{c_{t+1}}{k_{c,t+1}}+1-\delta\right)\left(\widehat{G}_{Z,t+1}e^{\epsilon_{Z,t+1}}\right)^{-(1-\alpha)} \\
\zeta\ell_{c,t} &= (1-\alpha)\left(1-\ell_t\right) \\
c_t &= k_{c,t}^{\alpha}\ell_{c,t}^{1-\alpha} \\
i_t &= k_{i,t}^{\alpha}\ell_{i,t}^{1-\alpha} \\
k_t &= (1-\delta)\frac{k_{t-1}}{G_H\widehat{G}_{A,t}e^{\epsilon_{A,t}}\widehat{G}_{Z,t}e^{\epsilon_{Z,t}}}+i_t \\
\frac{k_{t-1}}{G_H\widehat{G}_{A,t}e^{\epsilon_{A,t}}\widehat{G}_{Z,t}e^{\epsilon_{Z,t}}} &= k_{c,t}+k_{i,t}
\end{aligned}
$$

Learning process

Constant gain learning:

$$\ln \widehat{G}_{X,t} = \ln \widehat{G}_{X,t-1} + \lambda \left(\ln \left(\frac{X_t}{X_{t-1}} \right) - \ln \widehat{G}_{X,t-1} \right)$$

where

$$\lambda = \frac{\sigma_\nu^2}{\sigma_\epsilon^2} \left(-1 + \sqrt{1 + \frac{4\sigma_\epsilon^2}{\sigma_\nu^2}} \right)$$

In the paper $\lambda = 0.115$.
And,

$$\ln \widehat{G}_{X,t+1} = \ln \widehat{G}_{X,t} + \nu_{X,t+1}$$

SWZ model

The governement objective is

$$\min_{\{x_t\}_{t=0}^{\infty}} E \sum_{t=1}^{\infty} \delta^t \left[\left(\pi_t - \pi^\star\right)^2 + \lambda \left(u_t - u^{**}\right) \right]$$

and the dynamics of the economy is

$$\begin{aligned} u_t - u^\star &= \theta_0 \left(\pi_t - E_{t-1}\pi_t\right) + \theta_1 \left(\pi_{t-1} - E_{t-2}\pi_{t-1}\right) + \tau_1 \left(u_{t-1} - u^\star \right. \\ \pi_t &= x_{t-1} + \sigma_2 w_{2,t} \end{aligned}$$

π_t, inflation rate, $\pi^\star$, inflation target.
u_t, unemployment rate, $u^\star$, natural rate of unemployment,
$u^{\star\star}$, unemployment rate target.
x_{t-1}, policy instrument.
$w_{1,t}$ and $w_{2,t}$ are unit–variance, uncorrelated, white noises.

A misspecified Phillips curve

The government believes that unemployment is driven by

$$u_t = \alpha_0 \pi_t + \alpha_1 \pi_{t-1} + \alpha_2 u_{t-1} + \alpha_3 \pi_{t-2} + \alpha_4 u_{t-2} + \alpha_5 + \sigma w_t$$

The government is going to revise his estimates of the α coefficients, but will never learn the true model.

Government decision problem

In this later case, the Lagrangian is

$$\mathcal{L} = E\sum_{t=1}^{\infty}\delta^{t}\Big[(\pi_t - \pi^{\star})^2 + \lambda(u_t - u^{**})$$

$$+L_{1t}(x_{t-1} + \sigma_2 w_{2,t} - \pi_t) + L_{2t}(\alpha_0\pi_t + \alpha_1\pi_{t-1} + \alpha_2 u_{t-1} + \alpha$$

Dynamic model

The first order conditions together with the constraints define the dynamics of the system as seen by the government

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \pi_t} &= 2\left(\pi_t - \pi^\star\right) + \alpha_0 L_{1t} + \alpha_1 \delta L_{2t+1} + \alpha_3 \delta^2{}_{2t+2} \\
&= 0 \\
\frac{\partial \mathcal{L}}{\partial u_t} &= 2\left(u_t - u^{\star\star}\right) - \mu_{2t} + \alpha_2 \delta L_{2t+1} - \alpha_4 \delta^2 L_{2t+2} - L_{1t} \\
&= 0 \\
\frac{\partial \mathcal{L}}{\partial x_t} &= L_{1t+1} \\
&= 0 \\
u_t &= \alpha_0 \pi_t + \alpha_1 \pi_{t-1} + \alpha_2 u_{t-1} + \alpha_3 \pi_{t-2} + \alpha_4 u_{t-2} + \alpha_5 + \sigma w_t \\
\pi_t &= x_{t-1} + \sigma_2 w_{2t}
\end{aligned}
$$

Learning process

It is believed that the Phillips curve coefficients follow a random walk

$$
\begin{aligned}
u_t &= \alpha_t' \Phi_t + \sigma w_t \\
\alpha_t &= \alpha_{t-1} + \Lambda_t
\end{aligned}
$$

This, in turn, implies the following Kalman filter updating equations

$$
\begin{aligned}
\hat{\alpha}_{t+1|t} &= \hat{\alpha}_{t|t-1} + \frac{P_{t|t-1}\Phi_t \left(u_t - \Phi_t' \hat{\alpha}_{t|t-1}\right)}{\sigma^2 + \Phi_t' P_{t|t-1} \Phi_t} \\
P_{t+1|t} &= P_{t|t-1} + \frac{P_{t|t-1}\Phi_t \Phi_t' P_{t|t-1}}{\sigma^2 + \Phi_t' P_{t|t-1} \Phi_t} + V
\end{aligned}
$$

with $\alpha_{1|0} = \alpha_0$ and $P_{1|0} = P_0$ given and $V = \mathcal{E}\left(\Lambda_t \Lambda_t'\right)$.